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Global existence of classical solutions to the Cauchy problem on a semi-bounded initial axis for a nonhomogeneous quasilinear hyperbolic system

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Abstract

It is proven that if the leftmost eigenvalue is weakly linearly degenerate, then the Cauchy problem for a class of nonhomogeneous quasilinear hyperbolic systems with small and decaying initial data given on a semi-bounded axis admits a unique global C^1 solution on the domain $\{(t, x) \mid t \geq 0, x \geq x_n(t)\}$, where $x = x_n(t)$ is the fastest forward characteristic emanating from the origin. As an application of our result, we prove the existence of global classical, C^1 solutions of the flow equations of a model class of fluids with viscosity induced by fading memory with small smooth initial data given on a semi-bounded axis.

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1. Introduction and main result

Consider the following nonhomogeneous quasilinear system of equations:

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} + Lu = 0, \quad (1.1)$$

where $u = (u_1, \dots, u_n)^T$ is the unknown vector function of (t, x) , $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ ($i, j = 1, \dots, n$), $L > 0$ is a constant, system (1.1) is assumed to be hyperbolic, i.e., for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \dots, \lambda_n(u)$ and a complete set of left (respectively right) eigenvectors. Let $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ (respectively $r_i(u) = (r_{i1}(u), \dots, r_{in}(u))^T$) be a left (respectively right) eigenvector corresponding to $\lambda_i(u)$ ($i = 1, \dots, n$):

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{respectively } A(u)r_i(u) = \lambda_i(u)r_i(u)), \quad (1.2)$$

we have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{equivalently, } \det|r_{ij}(u)| \neq 0). \quad (1.3)$$

Without loss of generality, we may assume that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n), \quad (1.4)$$

where δ_{ij} stands for Kronecker's symbol.

We also assume that all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ ($i, j = 1, \dots, n$) have the same regularity as $a_{ij}(u)$ ($i, j = 1, \dots, n$).

We are interested in the global existence of classical, C^1 solutions to the Cauchy problem for system (1.1) with the following decay initial data:

$$t = 0: u = \varphi(x) \quad (x \geq 0), \quad (1.5)$$

where $\varphi(x)$ is a small C^1 vector function of x with certain decay properties as x tends to infinity.

For the case of $L = 0$, i.e., homogeneous quasilinear hyperbolic systems, such kinds of problems have been extensively studied by many authors in the literature, starting from the pioneering work of John [2] (for instance, see [2–16, 22, 34] and references therein). In particular, by introducing the concept of weak linear degeneracy, Li et al. [6–10, 14] gave a complete result on the global existence and life-span of classical, C^1 solutions to the Cauchy problem for the homogeneous quasilinear strictly hyperbolic systems, Li and Wang [11] also investigated the global existence of classical, C^1 solutions to the Cauchy problem on semi-bounded initial axis with small and decaying initial data. On the other hand, for the Cauchy problem of nonhomogeneous quasilinear hyperbolic systems, many results on the global existence of weak solutions have also been obtained by Liu, Chen, Dafermos, Hsiao and others (for instance, see [1, 17–21, 26–33] and references therein), and some methods have been established. So the following question arises naturally: In the nonhomogeneous case, can we obtain the global existence and uniqueness of classical, C^1 solutions to the Cauchy problem on semi-bounded initial axis? It is well known that this problem is of great importance from the viewpoint of both the development of the theory and the application. However, this problem is quite difficult, hence our work may provide

a simpler approach to this problem. Here, it should also be mentioned that in the early 1970s, Gu [16] proved that with the uniform damping term “ Lu ” in the equation, small smooth initial data would yield global continuous solutions without the weakly linear degeneracy hypothesis. By constructing an example, we [20] first illustrate that if system (1.1) is genuinely nonlinear, then the first derivatives of the solution $u = u(t, x)$ may blow up in a finite time even for arbitrary small and decaying C^1 initial data $\varphi(x)$ satisfying that there exists a constant $\mu > 0$ such that

$$\sup_{x \in \mathbb{R}} (1 + |x|)^{1+\mu} (|\varphi(x)| + |\varphi'(x)|) \ll 1$$

and this may be identified physically with the development of shock waves.

The main result in this paper is the following.

Theorem 1.1. *Suppose that in a neighborhood of $u = 0$, $A(u) \in C^2$ and*

$$\lambda_1(u), \dots, \lambda_{n-1}(u) < \lambda_n(u). \quad (1.6)$$

Suppose furthermore that system (1.1) is hyperbolic, $\lambda_n(u)$ is weakly linearly degenerate, i.e., along the n th characteristic trajectory $u = u^{(n)}(s)$ passing through $u = 0$, defined by

$$\begin{cases} \frac{du}{ds} = r_n(u), \\ s = 0: u = 0, \end{cases} \quad (1.7)$$

we have

$$\nabla \lambda_n(u) r_n(u) \equiv 0, \quad \forall |u| \text{ small}, \quad (1.8)$$

namely,

$$\lambda_n(u^{(n)}(s)) \equiv \lambda_n(0), \quad \forall |s| \text{ small}. \quad (1.9)$$

Suppose finally that $\varphi(x)$ is a C^1 vector function satisfying that there exists a constant $\mu > 0$ such that

$$\theta \triangleq \sup_{x \geq 0} (1 + |x|)^{1+\mu} (|\varphi(x)| + |\varphi'(x)|) < +\infty. \quad (1.10)$$

Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, Cauchy problem (1.1) and (1.5) admits a unique global C^1 solution $u = u(t, x)$ on the domain $D = \{(t, x) \mid t \geq 0, x \geq x_n(t)\}$, where $x = x_n(t)$ is the n th forward characteristic emanating from the origin $O(0, 0)$:

$$\begin{cases} \frac{dx_n(t)}{dt} = \lambda_n(u(t, x_n(t))), \\ x_n(0) = 0. \end{cases} \quad (1.11)$$

Remark 1.1. Suppose that in a neighborhood of $u = 0$,

$$\lambda_1(u), \dots, \lambda_p(u) < \lambda_{p+1}(u) \equiv \dots \equiv \lambda_n(u), \quad (1.12)$$

where $\lambda(u) \triangleq \lambda_{p+1}(u) \equiv \dots \equiv \lambda_n(u)$ is a characteristic with constant multiplicity $n - p$.

Suppose furthermore that $\lambda_{p+1}(u), \dots, \lambda_n(u)$ are weakly linearly degenerate (see [12]). Then the conclusion of Theorem 1.1 is still valid.

Remark 1.2. When

$$\lambda_1(0) < \lambda_2(0), \dots, \lambda_n(0) \quad (1.13)$$

or in a neighborhood of $u = 0$,

$$\lambda_1(u) \equiv \dots \equiv \lambda_p(u) < \lambda_{p+1}(u), \dots, \lambda_n(u), \quad (1.14)$$

for the initial data:

$$t = 0: u = \varphi(x) \quad (x \leq 0) \quad (1.15)$$

such that

$$\theta \triangleq \sup_{x \leq 0} (1 + |x|)^{1+\mu} (|\varphi(x)| + |\varphi'(x)|) < +\infty, \quad (1.16)$$

similar results hold as in Theorem 1.1 and Remark 1.1.

According to the local existence and uniqueness of the C^1 solutions to the Cauchy problem for nonhomogeneous quasilinear hyperbolic systems (see [24, Theorem VI]; see also [23,25]), in order to prove Theorem 1.1, it suffices to obtain a uniform a priori estimate on the C^0 norm of u and u_x on any given existence domain of the C^1 solution $u = u(t, x)$. We first study several formulas on the decomposition of waves for system (1.1), which will play an important role in our proof. Then, we prove the global existence and uniqueness of C^1 solutions to Cauchy problem (1.1) and (1.5) by establishing a uniform a priori estimate on the C^0 norm of u and u_x . The novelty of this paper is the presence of the nonhomogeneous term, although basic estimate methods are qualitatively the same as those [7,11–15] for homogeneous quasilinear hyperbolic systems, but new proofs are required, taking into account the more complicated uniform a priori estimate associated with the nonhomogeneous term. The rest of the paper is organized as follows. In Section 2, we give the main tools of the proof, that is several formulas on the decomposition of waves for the nonhomogeneous system (1.1). The main result will be proved in Section 3. An application is discussed in Section 4.

Finally, to show the applicability of our results, here we give an example of equations of type (1.1):

Example. The system of the flow equations of a model class of fluids with viscosity induced by fading memory (cf. [1])

$$\begin{cases} w_t - v_x + w = 0, \\ v_t - (\sigma(w))_x + v = 0, \end{cases} \quad (1.17)$$

where $\sigma(w)$ is a suitably smooth function of w such that $\sigma'(0) > 0$.

2. Preliminaries

Suppose that $A(u) \in C^k$, where k is an integer ≥ 1 . By [6, Lemma 2.5], there exists an invertible C^{k+1} transformation $u = u(\tilde{u})$ ($u(0) = 0$) such that in \tilde{u} -space, for each $i = 1, \dots, n$, the i th characteristic trajectory passing through $\tilde{u} = 0$ coincides with the \tilde{u}_i -axis at least for $|\tilde{u}_i|$ small, namely,

$$\tilde{r}_i(\tilde{u}_i e_i) \equiv e_i, \quad \forall |\tilde{u}_i| \text{ small } (i = 1, \dots, n), \quad (2.1)$$

where

$$e_i = (0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0)^T.$$

This transformation is called the normalized transformation, and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n)^T$ are called the normalized variables or normalized coordinates (see [7] or [11]).

Let

$$v_i = l_i(u)u \quad (i = 1, \dots, n), \quad (2.2)$$

and

$$w_i = l_i(u)u_x \quad (i = 1, \dots, n), \quad (2.3)$$

where

$$l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$$

denotes the i th left eigenvector.

By (1.4), it is easy to see that

$$u = \sum_{k=1}^n v_k r_k(u) \quad (2.4)$$

and

$$u_x = \sum_{k=1}^n w_k r_k(u). \quad (2.5)$$

Let

$$\frac{d}{d_t} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x} \quad (2.6)$$

be the directional derivative along the i th characteristic. Our aim in this section is to prove several formulas on the decomposition of waves for system (1.1), which will play an important role in our discussion.

Lemma 2.1 (Generalized John's formula).

$$\frac{d(e^{Lt} w_i)}{d_t} = \sum_{j,k=1}^n e^{Lt} \gamma_{ijk}(u) w_j w_k + \sum_{j,k=1}^n e^{Lt} \tilde{\gamma}_{ijk}(u) v_j w_k \quad (i = 1, \dots, n), \quad (2.7)$$

where

$$\gamma_{ijk}(u) = (\lambda_k(u) - \lambda_j(u)) r_j^T(u) \nabla l_i(u) r_k(u) - \nabla \lambda_i(u) r_j(u) \delta_{ik}, \quad (2.8)$$

$$\tilde{\gamma}_{ijk}(u) = -L r_j^T(u) \nabla l_i(u) r_k(u). \quad (2.9)$$

Hence, we have

$$\gamma_{ijj}(u) \equiv 0, \quad \forall j \neq i \quad (i, j = 1, \dots, n), \quad (2.10)$$

$$\gamma_{iii}(u) = -\nabla \lambda_i(u) r_i(u) \quad (i = 1, \dots, n). \quad (2.11)$$

Moreover, in the normalized coordinates,

$$\tilde{\gamma}_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \quad \forall i, j; \quad (2.12)$$

while, when the i th characteristic $\lambda_i(u)$ is weakly linearly degenerate, in the normalized coordinates,

$$\gamma_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \text{ small}, \quad \forall i. \quad (2.13)$$

Proof. By (2.3), (1.1) and (1.2), it is easy to see that

$$\begin{aligned} \frac{dw_i}{dt} &= u_i^T \nabla l_i(u) u_x + l_i(u) u_{xt} + \lambda_i(u) [u_x^T \nabla l_i(u) u_x + l_i(u) u_{xx}] \\ &= (-A(u) u_x - Lu)^T \nabla l_i(u) u_x + l_i(u) (-A(u) u_x - Lu)_x \\ &\quad + \lambda_i(u) [u_x^T \nabla l_i(u) u_x + l_i(u) u_{xx}] \\ &= -u_x^T A^T(u) \nabla l_i(u) u_x - Lu^T \nabla l_i(u) u_x - l_i(u) \frac{\partial}{\partial x} [A(u)] u_x - Lw_i \\ &\quad + \lambda_i(u) u_x^T \nabla l_i(u) u_x. \end{aligned} \quad (2.14)$$

Moreover, by (2.5), we have

$$u_x^T = \sum_{j=1}^n w_j r_j^T(u). \quad (2.15)$$

Thus, noting (2.5), (2.15) and the second equality in (1.2), it follows from (2.14) that

$$\begin{aligned} \frac{dw_i}{dt} &= \sum_{j,k=1}^n (\lambda_i(u) - \lambda_j(u)) r_j^T(u) \nabla l_i(u) r_k(u) w_j w_k - \sum_{k=1}^n l_i(u) \frac{\partial}{\partial x} [A(u)] r_k(u) w_k \\ &\quad - L \sum_{j,k=1}^n r_j^T(u) \nabla l_i(u) r_k(u) v_j w_k - Lw_i. \end{aligned} \quad (2.16)$$

Differentiating the first equality in (1.2) with respect to x yields

$$u_x^T \nabla l_i(u) A(u) + l_i(u) \frac{\partial}{\partial x} [A(u)] = \nabla \lambda_i(u) u_x l_i(u) + \lambda_i(u) u_x^T \nabla l_i(u). \quad (2.17)$$

Multiplying (2.17) by $r_k(u)$ and noting (1.4) and the second equality in (1.2), we get

$$l_i(u) \frac{\partial}{\partial x} [A(u)] r_k(u) = [\lambda_i(u) - \lambda_k(u)] u_x^T \nabla l_i(u) r_k(u) + \nabla \lambda_i(u) u_x \delta_{ik}. \quad (2.18)$$

Therefore, noting (2.5) and (2.15), it follows from (2.16) and (2.18) that

$$\begin{aligned} \frac{dw_i}{dt} &= \sum_{j,k=1}^n [(\lambda_k(u) - \lambda_j(u)) r_j^T(u) \nabla l_i(u) r_k(u) - \nabla \lambda_i(u) r_j(u) \delta_{ik}] w_j w_k \\ &\quad - L \sum_{j,k=1}^n r_j^T(u) \nabla l_i(u) r_k(u) v_j w_k - Lw_i. \end{aligned} \quad (2.19)$$

Hence, from (2.19) we immediately get (2.7).

Moreover, in the normalized coordinates, differentiating the relation $l_i(u_j e_j) r_j(u_j e_j) \equiv \delta_{ij}$ with respect to u_j and using (2.1), we get

$$r_j^T(u_j e_j) \nabla l_i(u_j e_j) r_j(u_j e_j) = -l_i(u_j e_j) \nabla r_j(u_j e_j) r_j(u_j e_j). \quad (2.20)$$

By (2.1), it is easy to see that (see also [7] or [11])

$$\tilde{\gamma}_{ijj}(u_j e_j) = L l_i(u_j e_j) \nabla r_j(u_j e_j) r_j(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \forall i, j. \quad (2.21)$$

This proves (2.12). \square

On the other hand, we have:

Lemma 2.2. For $i = 1, \dots, n$, it holds that

$$\frac{d(e^{Lt} v_i)}{d_i t} = \sum_{j,k=1}^n e^{Lt} \beta_{ijk}(u) v_j w_k + \sum_{j,k=1}^n e^{Lt} \tilde{\beta}_{ijk}(u) v_j v_k, \quad (2.22)$$

where

$$\beta_{ijk}(u) = (\lambda_i(u) - \lambda_k(u)) r_j^T(u) \nabla l_i(u) r_k(u), \quad (2.23)$$

$$\tilde{\beta}_{ijk}(u) = -L r_j^T(u) \nabla l_i(u) r_k(u). \quad (2.24)$$

Thus, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall i, j \ (i, j = 1, \dots, n). \quad (2.25)$$

Moreover, by (2.1), in the normalized coordinates we have

$$\beta_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \forall i, j, \quad (2.26)$$

and

$$\tilde{\beta}_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \forall i, j. \quad (2.27)$$

Proof. In a way completely similar to the proof of Lemma 2.1, we can prove (2.22)–(2.27) without any essential difficulty. Here we omit the details. \square

For any given $y \geq 0$, on the existence domain of C^1 solution, let $x = \tilde{x}_i(t, y)$ be the i th characteristic passing through point $(0, y)$:

$$\begin{cases} \frac{d\tilde{x}_i(t, y)}{dt} = \lambda_i(u(t, \tilde{x}_i(t, y))), \\ \tilde{x}_i(0, y) = y. \end{cases} \quad (2.28)$$

Lemma 2.3. Let $q_i(t, x)$ be defined by $q_i(t, \tilde{x}_i(t, y)) = w_i(t, \tilde{x}_i(t, y)) \frac{\partial \tilde{x}_i(t, y)}{\partial y}$, then along the i th characteristic $x = \tilde{x}_i(t, y)$ we have

$$\frac{d(e^{Lt} q_i)}{d_i t} = \sum_{j,k=1}^n e^{Lt} \Gamma_{ijk}(u) \frac{\partial \tilde{x}_i(t, y)}{\partial y} w_j w_k + \sum_{j,k=1}^n e^{Lt} \tilde{\gamma}_{ijk}(u) \frac{\partial \tilde{x}_i(t, y)}{\partial y} v_j w_k, \quad (2.29)$$

where $\tilde{\gamma}_{ijk}(u)$ is given by (2.9) and

$$\Gamma_{ijk}(u) = (\lambda_k(u) - \lambda_j(u)) r_j^T(u) \nabla l_i(u) r_k(u). \quad (2.30)$$

Hence,

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall i, j. \quad (2.31)$$

Proof. Differentiating the first equation of (2.28) with respect to y gives

$$\frac{d}{dt} \left(\frac{\partial \tilde{x}_i(t, y)}{\partial y} \right) = \nabla \lambda_i(u(t, \tilde{x}_i(t, y))) \frac{\partial u}{\partial x}(t, \tilde{x}_i(t, y)) \frac{\partial \tilde{x}_i(t, y)}{\partial y}. \quad (2.32)$$

Then, noting (2.7), it follows from (2.32) that

$$\begin{aligned} \frac{d(e^{Lt} q_i)}{d_i t} &= \frac{d(e^{Lt} w_i)}{d_i t} \frac{\partial \tilde{x}_i(t, y)}{\partial y} + e^{Lt} w_i \frac{d}{d_i t} \left(\frac{\partial \tilde{x}_i(t, y)}{\partial y} \right) \\ &= \left(\sum_{j,k=1}^n e^{Lt} [\gamma_{ijk}(u) w_j w_k + \tilde{\gamma}_{ijk}(u) v_j w_k] + e^{Lt} w_i \nabla \lambda_i(u) u_x \right) \frac{\partial \tilde{x}_i(t, y)}{\partial y}. \end{aligned} \quad (2.33)$$

Thus, from (2.5), (2.8) and (2.33), we immediately get (2.29)–(2.30). This completes the proof. \square

Similarly, noting (2.5), by (2.22) and (2.32), we have:

Lemma 2.4. Let $p_i(t, x)$ be defined by $p_i(t, \tilde{x}_i(t, y)) = v_i(t, \tilde{x}_i(t, y)) \frac{\partial \tilde{x}_i(t, y)}{\partial y}$, then along the i th characteristic $x = \tilde{x}_i(t, y)$ we have

$$\frac{d(e^{Lt} p_i)}{d_i t} = \sum_{j,k=1}^n e^{Lt} B_{ijk}(u) \frac{\partial \tilde{x}_i(t, y)}{\partial y} v_j w_k + \sum_{j,k=1}^n e^{Lt} \tilde{\beta}_{ijk}(u) \frac{\partial \tilde{x}_i(t, y)}{\partial y} v_j v_k, \quad (2.34)$$

where $\tilde{\beta}_{ijk}(u)$ is given by (2.24) and

$$B_{ijk}(u) = \beta_{ijk}(u) + \nabla \lambda_i(u) r_k(u) \delta_{ij}. \quad (2.35)$$

By (2.25), it is easy to see that

$$B_{iji}(u) \equiv 0, \quad \forall i \neq j \ (i, j = 1, \dots, n), \quad (2.36)$$

$$B_{iii}(u) = \nabla \lambda_i(u) r_i(u), \quad \forall i \ (i = 1, \dots, n). \quad (2.37)$$

Moreover, by (2.26), in the normalized coordinates we have

$$B_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \forall j \neq i; \quad (2.38)$$

while, when the i th characteristic $\lambda_i(u)$ is weakly linearly degenerate, in the normalized coordinates,

$$B_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \text{ small}, \forall i. \quad (2.39)$$

3. Proof of Theorem 1.1

In what follows, we always assume that $\theta > 0$ is suitably small.

By (1.6), there exist positive constants δ and δ_0 so small that

$$\lambda_n(u) - \lambda_i(v) \geq 2\delta_0, \quad \forall |u|, |v| \leq \delta \quad (i = 1, \dots, n-1), \quad (3.1)$$

and

$$|\lambda_i(u) - \lambda_i(v)| \leq \frac{\delta_0}{2}, \quad \forall |u|, |v| \leq \delta \quad (i = 1, \dots, n). \quad (3.2)$$

Without loss of generality, we may assume that

$$\lambda_i(0) \geq \delta_0 \quad (i = 1, \dots, n). \quad (3.3)$$

For the time being we assume that on the existence domain of the C^1 solution $u = u(t, x)$ to Cauchy problem (1.1) and (1.5), we have

$$|u(t, x)| \leq \delta. \quad (3.4)$$

At the end of the proof of Lemma 3.1, we will explain that this hypothesis is reasonable. Thus, in order to prove Theorem 1.1, we only need to establish a uniform a priori estimate on the C^0 norm of v and w defined by (2.2)–(2.3) on the existence domain of the C^1 solution $u = u(t, x)$.

By (3.2) and (3.3), it is easy to get

$$x_n(t) \geq \left(\lambda_n(0) - \frac{\delta_0}{2} \right) t \geq \frac{\delta_0}{2} t. \quad (3.5)$$

For any fixed $T > 0$, let

$$D^T = \{(t, x) \mid 0 \leq t \leq T, x \geq x_n(t)\}. \quad (3.6)$$

On each existence domain D^T of the C^1 solution $u = u(t, x)$, let

$$V_\infty^c(T) = \max_{i=1, \dots, n-1} \sup_{(t, x) \in D^T} \{(1+x)^{1+\mu} |v_i(t, x)|\}, \quad (3.7)$$

$$W_\infty^c(T) = \max_{i=1, \dots, n-1} \sup_{(t, x) \in D^T} \{(1+x)^{1+\mu} |w_i(t, x)|\}, \quad (3.8)$$

$$U_\infty^c(T) = \max_{i=1, \dots, n-1} \sup_{(t, x) \in D^T} \{(1+x)^{1+\mu} |u_i(t, x)|\}, \quad (3.9)$$

$$\tilde{V}_1(T) = \max_{j=1, \dots, n-1} \int_{C_j} |v_n(t, x)| dt, \quad (3.10)$$

$$\tilde{W}_1(T) = \max_{j=1, \dots, n-1} \int_{C_j} |w_n(t, x)| dt, \quad (3.11)$$

where C_j denotes any given j th characteristic in D^T ,

$$V_1(T) = \sup_{0 \leq t \leq T} \int_{D^T(t)} |v_n(t, x)| dx, \quad (3.12)$$

$$W_1(T) = \sup_{0 \leq t \leq T} \int_{D^T(t)} |w_n(t, x)| dx, \quad (3.13)$$

where $D^T(t)$ ($t \geq 0$) denotes the t -section of D^T :

$$D^T(t) = \{(\tau, x) \mid \tau = t, (\tau, x) \in D^T\}, \quad (3.14)$$

$$V_\infty(T) = \max_{i=1, \dots, n} \sup_{(t, x) \in D^T} |v_i(t, x)| \quad (3.15)$$

and

$$W_\infty(T) = \max_{i=1, \dots, n} \sup_{(t, x) \in D^T} |w_i(t, x)|. \quad (3.16)$$

Obviously, $V_\infty(T)$ is equivalent to

$$U_\infty(T) = \max_{i=1, \dots, n} \sup_{(t, x) \in D^T} |u_i(t, x)|. \quad (3.17)$$

Lemma 3.1. *Under the assumptions of Theorem 1.1, in the normalized coordinates there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t, x)$ to Cauchy problem (1.1) and (1.5), we have the following uniform a priori estimates:*

$$\tilde{W}_1(T), W_1(T), \tilde{V}_1(T), V_1(T) \leq k_1\theta, \quad (3.18)$$

$$W_\infty^c(T), V_\infty^c(T) \leq k_2\theta, \quad (3.19)$$

$$W_\infty(T), V_\infty(T), U_\infty(T) \leq k_3\theta \quad (3.20)$$

and

$$U_\infty^c(T) \leq k_4\theta, \quad (3.21)$$

here and henceforth k_i ($i = 1, 2, \dots$) are positive constants independent of θ and T .

Proof. We first estimate $\tilde{W}_1(T)$.

Passing through any fixed point $A(t_0, x_0) \in D^T$, we draw the j th characteristic $C_j: x = x_j(t)$ ($0 \leq t \leq t_0, j \neq n$) which intersects the x -axis at a point $C(0, y_2)$. In the meantime, passing through any given point $(t, x_j(t))$ on C_j , we draw the n th characteristic $\xi = x_n(s, y)$ which intersects the x -axis at a point $B_y(0, y)$, see Fig. 1.

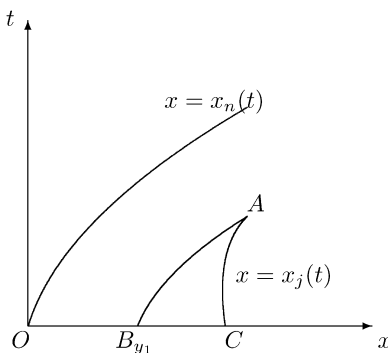


Fig. 1.

Obviously, we have

$$x_n(t, y) = x_j(t) \quad (3.22)$$

which gives a one-to-one correspondence $t = t(y)$ between the segment $\overline{CB_{y_1}}$ and C_j ($0 \leq t \leq t_0$). Thus, the integral on C_j with respect to t can be reduced to the integral with respect to y . Differentiating (3.22) with respect to t gives

$$dt = \frac{1}{\lambda_j(u(t, x_n(t, y))) - \lambda_n(u(t, x_n(t, y)))} \frac{\partial x_n(t, y)}{\partial y} dy, \quad (3.23)$$

in which $t = t(y)$. Thus, noting (3.1) and (3.4) it is easy to see that in order to estimate

$$\int_{C_j} |w_n(t, x)| dt = \int_0^{t_0} |w_n(t, x_j(t))| dt = \int_0^{t_0} |w_n(t, x_n(t, y))| dt, \quad (3.24)$$

it suffices to estimate

$$\int_{y_1}^{y_2} |q_n(t, x_n(t, y))|_{t=t(y)} dy. \quad (3.25)$$

Integrating (2.29) along $\xi = x_n(s, y)$ and noting that $\frac{\partial x_n(0, y)}{\partial y} \equiv 1$ yields

$$\begin{aligned} & q_n(t, x_n(t, y))|_{t=t(y)} \\ &= e^{-L t(y)} w_n(0, y) + \int_0^{t(y)} e^{-L(t(y)-s)} \frac{\partial x_n(s, y)}{\partial y} \\ & \quad \times \left[\sum_{j,k=1}^n \Gamma_{njk}(u) w_j w_k + \sum_{j,k=1}^n \tilde{\gamma}_{njk}(u) v_j w_k \right] (s, x_n(s, y)) ds. \end{aligned} \quad (3.26)$$

Noting (2.12) and (2.31), (3.26) can be rewritten as

$$\begin{aligned} & q_n(t, x_n(t, y))|_{t=t(y)} \\ &= e^{-L t(y)} w_n(0, y) + \int_0^{t(y)} e^{-L(t(y)-s)} \frac{\partial x_n(s, y)}{\partial y} \left[(\tilde{\gamma}_{nnn}(u) - \tilde{\gamma}_{nnn}(u_n e_n)) v_n w_n \right. \\ & \quad + \sum_{j=1}^{n-1} (\tilde{\gamma}_{njn}(u) v_j w_n + \tilde{\gamma}_{nnj}(u) v_n w_j) + \sum_{j,k=1}^{n-1} \tilde{\gamma}_{njk}(u) v_j w_k \\ & \quad \left. + \sum_{j=1}^{n-1} (\Gamma_{njn}(u) + \Gamma_{nnj}(u)) w_j w_n + \sum_{j,k=1}^{n-1} \Gamma_{njk}(u) w_j w_k \right] (s, x_n(s, y)) ds. \end{aligned} \quad (3.27)$$

By Hadamard's formula, we get

$$\tilde{\gamma}_{nnn}(u) - \tilde{\gamma}_{nnn}(u_n e_n) = \sum_{l=1}^{n-1} \left(\int_0^1 \frac{\partial \tilde{\gamma}_{nnn}}{\partial u_l} (\tau u_1, \dots, \tau u_{n-1}, u_n) d\tau \right) u_l. \quad (3.28)$$

Noting (3.4), $L > 0$ and $\frac{\partial x_n(s, y)}{\partial y} > 0$, it follows from (2.27) and (3.28) that

$$\begin{aligned}
 & |q_n(t, x_n(t, y))|_{t=t(y)} \\
 & \leq |w_n(0, y)| + C_1 \left\{ [(W_\infty^c(T))^2 + W_\infty^c(T)V_\infty^c(T)] \right. \\
 & \quad \times \int_0^{t(y)} (1 + |x_n(s, y)|)^{-2(1+\mu)} \frac{\partial x_n(s, y)}{\partial y} ds + [W_\infty^c(T) + V_\infty^c(T) + U_\infty^c(T)V_\infty(T)] \\
 & \quad \times \int_0^{t(y)} (1 + |x_n(s, y)|)^{-(1+\mu)} |w_n(s, x_n(s, y))| \frac{\partial x_n(s, y)}{\partial y} ds \\
 & \quad \left. + W_\infty^c(T) \int_0^{t(y)} (1 + |x_n(s, y)|)^{-(1+\mu)} |v_n(s, x_n(s, y))| \frac{\partial x_n(s, y)}{\partial y} ds \right\}, \quad (3.29)
 \end{aligned}$$

here and henceforth C_i ($i = 1, 2, \dots$) will denote positive constants independent of θ and T . Noting that the transformation

$$\begin{cases} x = x_n(s, y), \\ t = s, \end{cases}$$

gives the area element

$$dt dx = \frac{\partial x_n(s, y)}{\partial y} ds dy, \quad (3.30)$$

it easily follows from (3.29) that

$$\begin{aligned}
 & \int_{y_1}^{y_2} |q_n(t, x_n(t, y))|_{t=t(y)} dy \\
 & \leq C_2 \left\{ \int_0^{+\infty} |w_n(0, y)| dy + W_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T)] \iint_{ABC} (1+x)^{-2(1+\mu)} dt dx \right. \\
 & \quad + [W_\infty^c(T) + V_\infty^c(T) + U_\infty^c(T)V_\infty(T)] \iint_{ABC} (1+x)^{-(1+\mu)} |w_n(t, x)| dt dx \\
 & \quad \left. + W_\infty^c(T) \iint_{ABC} (1+x)^{-(1+\mu)} |v_n(t, x)| dt dx \right\}. \quad (3.31)
 \end{aligned}$$

Moreover, by (3.5), it is easy to see that

$$x \geq x_n(t) \geq \frac{\delta_0}{2}t, \quad \forall (t, x) \in D^T.$$

Then, we have

$$\begin{aligned} \iint_{ABC} (1+x)^{-(1+\mu)} |w_n(t, x)| dt dx &\leq C_3 \iint_{ABC} (1+t)^{-(1+\mu)} |w_n(t, x)| dt dx \\ &\leq C_4 \left\{ W_1(T) \int_0^{+\infty} (1+t)^{-(1+\mu)} dt \right\} \leq C_5 W_1(T). \end{aligned}$$

Similarly, we have

$$\iint_{ABC} (1+x)^{-(1+\mu)} |v_n(t, x)| dt dx \leq C_6 V_1(T).$$

Thus, we get

$$\begin{aligned} \tilde{W}_1(T) &\leq C_7 \{ \theta + W_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T) + W_1(T) + V_1(T)] \\ &\quad + W_1(T) [V_\infty^c(T) + U_\infty^c(T) V_\infty(T)] \}. \end{aligned} \quad (3.32)$$

We next estimate $W_1(T)$.

Passing through any given point $(t, x) \in D^T(t)$ we draw the n th characteristic $\xi = x_n(s, y)$ which intersects the x -axis at a point $A_y(0, y)$. Obviously, we have $x = x_n(t, y)$. Thus, we get

$$\int_{x_n(t)}^a |w_n(t, x)| dx = \int_0^{y_1} |q_n(t, x_n(t, y))| dy, \quad (3.33)$$

where $A(0, y_1)$ denotes the intersection point of the n th characteristic passing through point $B(t, a)$ ($a > x_n(t)$) with x -axis (see Fig. 2).

Similarly to (3.31), it is easy get

$$\begin{aligned} &\int_{x_n(t)}^a |w_n(t, x)| dx \\ &\leq C_8 \left\{ \int_0^{+\infty} |w_n(0, y)| dy + W_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T)] \iint_{ABCO} (1+x)^{-2(1+\mu)} dt dx \right. \\ &\quad + [W_\infty^c(T) + V_\infty^c(T) + U_\infty^c(T) V_\infty(T)] \iint_{ABCO} (1+x)^{-(1+\mu)} |w_n(t, x)| dt dx \\ &\quad \left. + W_\infty^c(T) \iint_{ABCO} (1+x)^{-(1+\mu)} |v_n(t, x)| dt dx \right\} \\ &\leq C_9 \{ \theta + W_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T) + W_1(T) + V_1(T)] \\ &\quad + W_1(T) [V_\infty^c(T) + U_\infty^c(T) V_\infty(T)] \}. \end{aligned} \quad (3.34)$$

Letting $a \rightarrow +\infty$, we immediately get

$$\begin{aligned} W_1(T) &\leq C_9 \{ \theta + W_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T) + W_1(T) + V_1(T)] \\ &\quad + W_1(T) [V_\infty^c(T) + U_\infty^c(T) V_\infty(T)] \}. \end{aligned} \quad (3.35)$$

We next estimate $W_\infty^c(T)$.

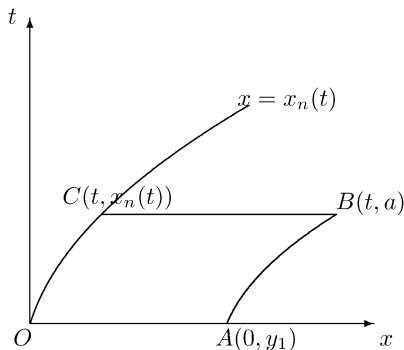


Fig. 2.

Passing through any given point $(t, x) \in D^T$ we draw the i th characteristic c_i : $\xi = x_i(s, y)$ ($0 \leq s \leq t, i \neq n$) which intersects the x -axis at a point $(0, y)$. Noting (3.3), by (3.2) we get

$$y \leq x_i(s, y) \leq y + \left(\lambda_i(0) + \frac{\delta_0}{2} \right) s, \quad \forall s \in [0, t] \ (i \neq n). \quad (3.36)$$

By (3.5), it is easy to see that

$$s \leq t \leq t_0, \quad (3.37)$$

where t_0 denotes the t -coordinate of the intersection point of the straight line $x = (\lambda_n(0) - \frac{\delta_0}{2})t$ with the straight line $x = y + (\lambda_i(0) + \frac{\delta_0}{2})t$ passing through the point $(0, y)$. Obviously,

$$t_0 = \frac{y}{\lambda_n(0) - \lambda_i(0) - \delta_0}. \quad (3.38)$$

Thus, it follows from (3.36) and (3.38) that

$$y \leq x_i(s, y) \leq \frac{\lambda_n(0) - \frac{\delta_0}{2}}{\lambda_n(0) - \lambda_i(0) - \delta_0} y, \quad \forall s \in [0, t] \ (i \neq n). \quad (3.39)$$

Integrating (2.7) along this characteristic gives

$$\begin{aligned} w_i(t, x) &= e^{-Lt} w_i(0, y) \\ &+ \int_0^t e^{-L(t-s)} \sum_{j,k=1}^n [\gamma_{ijk}(u) w_j w_k + \tilde{\gamma}_{ijk}(u) v_j w_k] (s, x_i(s, y)) ds. \end{aligned} \quad (3.40)$$

Noting (2.10) and (2.12), (3.40) can be rewritten as

$$\begin{aligned} w_i(t, x) &= e^{-Lt} w_i(0, y) + \int_0^t e^{-L(t-s)} \left[(\tilde{\gamma}_{inn}(u) - \tilde{\gamma}_{inn}(u_n e_n)) v_n w_n \right. \\ &+ \sum_{j,k=1}^{n-1} (\gamma_{ijk}(u) w_j w_k + \tilde{\gamma}_{ijk}(u) v_j w_k) + \sum_{j=1}^{n-1} ((\gamma_{ijn}(u) + \gamma_{inj}(u)) w_j w_n \\ &\left. + \tilde{\gamma}_{ijn}(u) v_j w_n + \tilde{\gamma}_{inj}(u) v_n w_j) \right] (s, x_i(s, y)) ds. \end{aligned} \quad (3.41)$$

Thus, noting that $L > 0$, and using (3.37)–(3.39) and Hadamard's formula, it follows from (3.41) that

$$\begin{aligned} (1+x)^{1+\mu} |w_i(t, x)| &\leq C_{10} \{ (1+y)^{1+\mu} |w_i(0, y)| + (1+y)^{-(1+\mu)} t [(W_\infty^c(T))^2 + W_\infty^c(T) V_\infty^c(T)] \\ &\quad + W_\infty^c(T) [\tilde{W}_1(T) + \tilde{V}_1(T)] + \tilde{W}_1(T) [V_\infty^c(T) + U_\infty^c(T) V_\infty(T)] \} \\ &\leq C_{11} \{ \theta + (1+y)^{-\mu} [(W_\infty^c(T))^2 + W_\infty^c(T) V_\infty^c(T)] \\ &\quad + W_\infty^c(T) [\tilde{W}_1(T) + \tilde{V}_1(T)] + \tilde{W}_1(T) [V_\infty^c(T) + U_\infty^c(T) V_\infty(T)] \}. \end{aligned} \quad (3.42)$$

Thus, we get

$$\begin{aligned} W_\infty^c(T) &\leq C_{11} \{ \theta + W_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T) + \tilde{W}_1(T) + \tilde{V}_1(T)] \\ &\quad + \tilde{W}_1(T) [V_\infty^c(T) + U_\infty^c(T) V_\infty(T)] \}. \end{aligned} \quad (3.43)$$

We next estimate $\tilde{V}_1(T)$ and $V_1(T)$.

Since $\lambda_n(u)$ is weakly linearly degenerate, by (2.39) we get

$$B_{nnn}(u_n e_n) \equiv 0. \quad (3.44)$$

Similarly to (3.24), in order to estimate $\tilde{V}_1(T)$ it suffices to estimate

$$\int_{y_1}^{y_2} |p_n(t, x_n(t, y))|_{t=t(y)} dy. \quad (3.45)$$

Since $\frac{\partial x_n(0, y)}{\partial y} \equiv 1$, integrating (2.34) along $\xi = x_n(s, y)$, and noting (2.36), (2.27) and (3.44), we get

$$\begin{aligned} p_n(t, x_n(t, y))|_{t=t(y)} &= e^{-L t(y)} v_n(0, y) + \int_0^{t(y)} e^{-L(t(y)-s)} \frac{\partial x_n(s, y)}{\partial y} \left[(\tilde{\beta}_{nnn}(u) - \tilde{\beta}_{nnn}(u_n e_n)) (v_n)^2 \right. \\ &\quad + \sum_{j=1}^{n-1} (\tilde{\beta}_{n j n}(u) + \tilde{\beta}_{n n j}(u)) v_n v_j + \sum_{j,k=1}^{n-1} \tilde{\beta}_{n j k}(u) v_j v_k + \sum_{j=1}^{n-1} B_{n n j}(u) w_j v_n \\ &\quad \left. + \sum_{j,k=1}^{n-1} B_{n j k}(u) v_j w_k + (B_{nnn}(u) - B_{nnn}(u_n e_n)) v_n w_n \right] (s, x_n(s, y)) ds. \end{aligned} \quad (3.46)$$

In a way similar to the proof of (3.31), we get from (3.46) that

$$\begin{aligned} \tilde{V}_1(T) &\leq C_{12} \left\{ \int_0^{+\infty} |v_n(0, y)| dy + V_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T)] \int_{ABC} \int_{ABC} (1+x)^{-2(1+\mu)} dt dx \right. \\ &\quad + [W_\infty^c(T) + V_\infty^c(T) + U_\infty^c(T) (W_\infty(T) + V_\infty(T))] \\ &\quad \left. \times \int_{ABC} \int_{ABC} (1+x)^{-(1+\mu)} |v_n(t, x)| dt dx \right\} \end{aligned}$$

$$\leq C_{13}\{\theta + V_{\infty}^c(T)[W_{\infty}^c(T) + V_{\infty}^c(T)] + [W_{\infty}^c(T) + V_{\infty}^c(T) + U_{\infty}^c(T)(W_{\infty}(T) + V_{\infty}(T))]V_1(T)\}. \quad (3.47)$$

Similarly to (3.35), we have

$$V_1(T) \leq C_{14}\{\theta + V_{\infty}^c(T)[W_{\infty}^c(T) + V_{\infty}^c(T)] + [W_{\infty}^c(T) + V_{\infty}^c(T) + U_{\infty}^c(T)(W_{\infty}(T) + V_{\infty}(T))]V_1(T)\}. \quad (3.48)$$

We next estimate $V_{\infty}^c(T)$.

Similarly to (3.41), we have

$$\begin{aligned} v_i(t, x) = & e^{-Lt}v_i(0, y) + \int_0^t e^{-L(t-s)} \left[(\beta_{inn}(u) - \tilde{\beta}_{inn}(u_n e_n))v_n w_n \right. \\ & + \sum_{j,k=1}^{n-1} (\beta_{ijk}(u)v_j w_k + \tilde{\beta}_{ijk}(u)v_j v_k) \\ & + \sum_{j=1}^{n-1} (\beta_{ijn}(u)v_j w_n + \beta_{inj}(u)w_j v_n + (\tilde{\beta}_{ijn}(u) + \tilde{\beta}_{inj}(u))v_n v_j) \\ & \left. + (\tilde{\beta}_{inn}(u) - \tilde{\beta}_{inn}(u_n e_n))(v_n)^2 \right] (s, x_i(s, y)) ds \quad (i \neq n). \end{aligned} \quad (3.49)$$

Thus, noting (3.37)–(3.39), and using Hadamard's formula, similarly to (3.42), we get

$$V_{\infty}^c(T) \leq C_{15}\{\theta + V_{\infty}^c(T)[W_{\infty}^c(T) + V_{\infty}^c(T) + \tilde{W}_1(T) + \tilde{V}_1(T)] + U_{\infty}^c(T)[\tilde{W}_1(T) + \tilde{V}_1(T)]V_{\infty}(T) + W_{\infty}^c(T)\tilde{V}_1(T)\}. \quad (3.50)$$

We next estimate $U_{\infty}^c(T)$.

For any given point $(t, x) \in D^T$, using (2.4) and noting (2.1), we have

$$\begin{aligned} u_i(t, x) = & \sum_{k=1}^n v_k r_k^T(u) e_i = \sum_{k=1}^{n-1} v_k r_k^T(u) e_i + v_n (r_n^T(u) - r_n^T(u_n e_n)) e_i \\ = & \sum_{k=1}^{n-1} v_k r_k^T(u) e_i + \sum_{l=1}^{n-1} v_n u_l a_{nl}(u) e_i \quad (i \neq n), \end{aligned} \quad (3.51)$$

where

$$a_{nl}(u) = \int_0^1 \frac{\partial r_n^T}{\partial u_l}(su_1, \dots, su_{n-1}, u_n) ds. \quad (3.52)$$

Thus, it is easy get

$$U_{\infty}^c(T) \leq C_{16}\{V_{\infty}^c(T) + U_{\infty}^c(T)V_{\infty}(T)\}. \quad (3.53)$$

We now estimate $V_{\infty}(T)$.

Passing through any given point $(t, x) \in D^T$ we draw the n th characteristic c_n : $\xi = x_n(s, y)$ ($0 \leq s \leq t$) which intersects the x -axis at a point $(0, y)$. Integrating (2.22) (in which we take $i = n$) along this characteristic yields

$$\begin{aligned} v_n(t, x) = & e^{-Lt} v_n(0, y) + \int_0^t e^{-L(t-s)} \left[\sum_{j,k=1}^{n-1} (\beta_{nj k}(u) v_j w_k + \tilde{\beta}_{nj k}(u) v_j v_k) \right. \\ & + \sum_{j=1}^{n-1} (\beta_{nnj}(u) w_j v_n + (\tilde{\beta}_{nnj}(u) + \tilde{\beta}_{nnj}(u)) v_j v_n) \\ & \left. + (\tilde{\beta}_{nnn}(u) - \tilde{\beta}_{nnn}(u_n e_n)) (v_n)^2 \right] (s, x_n(s, y)) ds. \end{aligned} \quad (3.54)$$

Then, noting (1.10) and (3.5), and using Hadamard's formula, we get

$$\begin{aligned} |v_n(t, x)| \leq & C_{17} \{ \theta + V_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T) + V_\infty(T)] \\ & + U_\infty^c(T) (V_\infty(T))^2 + W_\infty^c(T) V_\infty(T) \}. \end{aligned} \quad (3.55)$$

Hence, noting that $|v_i(t, x)|$ ($i = 1, \dots, n-1$) can be controlled by $V_\infty^c(T)$, we have

$$\begin{aligned} V_\infty(T) \leq & C_{18} \{ \theta + V_\infty^c(T) + V_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T) + V_\infty(T)] \\ & + U_\infty^c(T) (V_\infty(T))^2 + W_\infty^c(T) V_\infty(T) \}. \end{aligned} \quad (3.56)$$

We finally estimate $W_\infty(T)$.

Since $\lambda_n(u)$ is weakly linearly degenerate, by (2.13) we get

$$\gamma_{nnn}(u_n e_n) \equiv 0. \quad (3.57)$$

Similarly to (3.54), integrating (2.7) (in which we take $i = n$) along c_n gives

$$\begin{aligned} w_n(t, x) = & e^{-Lt} w_n(0, y) + \int_0^t e^{-L(t-s)} \left[\sum_{j,k=1}^{n-1} (\gamma_{nj k}(u) w_j w_k + \tilde{\gamma}_{nj k}(u) v_j w_k) \right. \\ & + \sum_{j=1}^{n-1} ((\gamma_{nnj}(u) + \gamma_{nnj}(u)) w_j w_n + \tilde{\gamma}_{nnj}(u) v_j w_n + \tilde{\gamma}_{nnj}(u) v_n w_j) \\ & + (\gamma_{nnn}(u) - \gamma_{nnn}(u_n e_n)) (w_n)^2 \\ & \left. + (\tilde{\gamma}_{nnn}(u) - \tilde{\gamma}_{nnn}(u_n e_n)) v_n w_n \right] (s, x_n(s, y)) ds. \end{aligned} \quad (3.58)$$

Similarly to (3.56), we have

$$\begin{aligned} W_\infty(T) \leq & C_{19} \{ \theta + W_\infty^c(T) + W_\infty^c(T) [W_\infty^c(T) + V_\infty^c(T) + W_\infty(T) + V_\infty(T)] \\ & + U_\infty^c(T) W_\infty(T) [W_\infty(T) + V_\infty(T)] + V_\infty^c(T) W_\infty(T) \}. \end{aligned} \quad (3.59)$$

We now prove (3.18)–(3.21).

Noting (1.10), evidently we have

$$W_\infty^c(0), V_\infty^c(0), U_\infty^c(0) \leq C_{20}\theta,$$

$$W_1(0) = V_1(0) = \tilde{W}_1(0) = \tilde{V}_1(0) = 0$$

and

$$W_\infty(0), V_\infty(0) \leq C_{21}\theta.$$

Thus, by continuity there exist positive constants k_1, k_2, k_3 and k_4 independent of θ , such that (3.18)–(3.21) hold at least for $0 \leq T \leq \tau_0$, where τ_0 is a small positive number. Hence, in order to prove (3.18)–(3.21) it suffices to show that we can choose k_1, k_2, k_3 and k_4 in such a way that for any fixed T_0 ($0 < T_0 \leq T$) such that

$$\tilde{W}_1(T_0), W_1(T_0), \tilde{V}_1(T_0), V_1(T_0) \leq 2k_1\theta, \quad (3.60)$$

$$W_\infty^c(T_0), V_\infty^c(T_0) \leq 2k_2\theta, \quad (3.61)$$

$$W_\infty(T_0), V_\infty(T_0) \leq 2k_3\theta, \quad (3.62)$$

$$U_\infty^c(T_0) \leq 2k_4\theta, \quad (3.63)$$

we have

$$\tilde{W}_1(T_0), W_1(T_0), \tilde{V}_1(T_0), V_1(T_0) \leq k_1\theta, \quad (3.64)$$

$$W_\infty^c(T_0), V_\infty^c(T_0) \leq k_2\theta, \quad (3.65)$$

$$W_\infty(T_0), V_\infty(T_0) \leq k_3\theta, \quad (3.66)$$

$$U_\infty^c(T_0) \leq k_4\theta. \quad (3.67)$$

Substituting (3.60)–(3.63) into the right-hand side of (3.32), (3.35), (3.43), (3.47)–(3.48), (3.50), (3.53), (3.56) and (3.59) (in which we take $T = T_0$), it is easy to see that, when $\theta_0 > 0$ is suitably small, we have

$$\tilde{W}_1(T_0) \leq 2C_7\theta, \quad (3.68)$$

$$W_1(T_0) \leq 2C_9\theta, \quad (3.69)$$

$$W_\infty^c(T_0) \leq 2C_{11}\theta, \quad (3.70)$$

$$\tilde{V}_1(T_0) \leq 2C_{13}\theta, \quad (3.71)$$

$$V_1(T_0) \leq 2C_{14}\theta, \quad (3.72)$$

$$V_\infty^c(T_0) \leq 2C_{15}\theta, \quad (3.73)$$

$$U_\infty^c(T_0) \leq 3C_{16}k_2\theta, \quad (3.74)$$

$$V_\infty(T_0) \leq 2C_{18}(1 + k_2)\theta, \quad (3.75)$$

$$W_\infty(T_0) \leq 2C_{19}(1 + k_2)\theta. \quad (3.76)$$

Hence, if $k_1 \geq 2 \max\{C_7, C_9, C_{13}, C_{14}\}$, $k_2 \geq 2 \max\{C_{11}, C_{15}\}$, $k_3 \geq 2(1 + k_2) \max\{C_{18}, C_{19}\}$ and $k_4 \geq 3C_{16}k_2$, then we get (3.64)–(3.67). This proves (3.18)–(3.21).

Finally, we point out that when $\theta_0 > 0$ is suitably small, by (3.20) we have

$$U_\infty(T) \leq k_3\theta \leq k_3\theta_0 \leq \frac{\delta}{2}.$$

This implies the validity of hypothesis (3.4). The proof of Lemma 3.1 is finished. \square

Proof of Theorem 1.1. It suffices to prove Theorem 1.1 in the normalized coordinates. Under the assumptions of Theorem 1.1, by (3.20) we know that if $\theta_0 > 0$ is suitably small, then for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $\{(t, x) \mid 0 \leq t \leq T, x \geq x_n(t)\}$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1) and (1.5), we have the following uniform a priori estimate on the C^1 norm of the solution:

$$\|u(t, \cdot)\|_{C^1} \triangleq \|u(t, \cdot)\|_{C^0} + \|u_x(t, \cdot)\|_{C^0} \leq k_5 \theta. \quad (3.77)$$

Thus we immediately get the conclusion of Theorem 1.1. \square

4. Application

Consider the following Cauchy problem for the system of the flow equations of a model class of fluids with viscosity induced by fading memory (cf. [1,17,18]):

$$\begin{cases} w_t - v_x + w = 0, \\ v_t - (\sigma(w))_x + v = 0, \end{cases} \quad (4.1)$$

$$t = 0: (w, v) = (w_0(x), \tilde{v}_0 + v_0(x)) \quad (x \geq 0), \quad (4.2)$$

where $\sigma(w)$ is a suitably smooth function of w such that

$$\sigma'(0) > 0, \quad (4.3)$$

\tilde{v}_0 is a constant, and $w_0(x), v_0(x)$ are C^1 functions satisfying the decaying property as shown in (1.10).

Let

$$u = \begin{pmatrix} w \\ v \end{pmatrix}. \quad (4.4)$$

By (4.3), it is easy to see that in a neighborhood of $u_0 = \begin{pmatrix} 0 \\ \tilde{v}_0 \end{pmatrix}$, system (4.1) is strictly hyperbolic and has the following two distinct real eigenvalues:

$$\lambda_1(u) \triangleq -\sqrt{\sigma'(w)} < \lambda_2(u) \triangleq \sqrt{\sigma'(w)}. \quad (4.5)$$

The corresponding right and left eigenvectors are

$$r_1(u) \parallel (1, \sqrt{\sigma'(w)})^T, \quad r_2(u) \parallel (1, -\sqrt{\sigma'(w)})^T \quad (4.6)$$

and

$$l_1(u) \parallel (\sqrt{\sigma'(w)}, 1), \quad l_2(u) \parallel (-\sqrt{\sigma'(w)}, 1), \quad (4.7)$$

respectively. It is easy to see that in a neighborhood of u_0 , all characteristics are linearly degenerate, then weakly linearly degenerate, provided that

$$\sigma''(w) \equiv 0, \quad \forall |w| \text{ small}. \quad (4.8)$$

By Theorem 1.1 we get:

Theorem 4.1. Suppose that (4.8) holds. Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, Cauchy problem (4.1) and (4.2) admits a unique global C^1 solution $u = u(t, x)$ on the domain $D = \{(t, x) \mid t \geq 0, x \geq x_2(t)\}$, where $x = x_2(t)$ is the second forward characteristic emanating from origin $O(0, 0)$.

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